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INTEGRAL TREATMENT OF NEUTRON DIFFUSION

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Integral Equation

The distribution of neutrons is described by the integral equation

$$N(r,t) = \int \frac{1+f}{4\pi} \frac{\sigma_t \, d\vec{r}'}{|r-r'|^2} N\left(r', t - \frac{|r-r'|}{v}\right) e^{-\sigma_t |r-r'|}$$

where σ_t is the total cross section per unit length, v the neutron velocity, $1 + f$ the mean number of neutrons emerging per collision. Here two simplifications have been used. All collision processes have been assumed isotropic and the neutrons assumed monochromatic. Here f may be dependent on position but not σ_t. Corrections for anisotropy of scattering will be discussed later.

We look for a solution of the form

$$N(r,t) = N(r)\, e^{\gamma t}$$

then

$$N(r) = \int \frac{(1+f)}{4\pi} \frac{\sigma_t \, dr'}{|r-r'|^2} e^{-(\sigma_t + \frac{\gamma}{v})|r-r'|} N(r')$$

Simplify by taking unit of time and distance such that $\sigma_t = v = 1$, i.e., the mean free path and time are units of distance and time.

$$N(r) = \int \frac{1+f}{4\pi} \frac{dr'}{|r-r'|^2} e^{-(1+\gamma)|r-r'|} N(r')$$

introduce $R = (1+\gamma)r$ giving

$$N(R) = \int \frac{1+f}{4\pi(1+\gamma)} \frac{dR'}{|R-R'|^2} e^{-|R-R'|} N(R')$$

1. Consider first plane slab, tamped or untamped. Then have

$$N(R) = N(x)$$

$$N(x) = \int \frac{1+f}{4\pi(1+\gamma)} \frac{dx'\, 2\pi\rho\, d\rho}{\left((x-x')^2+\rho^2\right)} e^{-\sqrt{(x-x')^2+\rho^2}} N(x')$$

$$\sqrt{(x-x')^2+\rho^2} = \ell \;; \qquad \rho\, d\rho = \ell\, d\ell$$

$$N(x) = \int \frac{1+f}{2(1+\gamma)} dx' \int_{|x-x'|}^{\infty} \frac{d\ell}{\ell} e^{-\ell} N(x')$$

$$= \int dx' \frac{1+f}{2(1+\gamma)} E_i(|x-x'|) N(x')$$

(2) Suppose spherical, then $N(\vec{R}) = N(r)$

$$N(r) = \int \frac{1+f}{4\pi(1+\gamma)} \frac{r'^2 dr'\, d\varphi\, d\mu}{r^2 + r'^2 - 2rr'\mu} e^{-\sqrt{r^2+r'^2-2rr'\mu}} N(r')$$

write $N(r) = rU(r)$

$$U(r) = \int_0^{\infty} \frac{1+f}{2(1+\gamma)} dr' \int_{-1}^{1} \frac{d\mu\, r'r\, e^{-\sqrt{r^2+r'^2-2rr'\mu}}}{r^2+r'^2-2rr'\mu} U(r')$$

$$\ell = \sqrt{r^2+r'^2-2rr'\mu} \;, \quad rr'd\mu = -\tfrac{1}{2} d\ell^2$$

$$\therefore \int_{-1}^{1} \frac{d\mu\, rr' e^{-\ell}}{\ell^2} = -\tfrac{1}{2}\int \frac{d\ell^2}{\ell^2} e^{-\ell} = -\int \frac{d\ell}{\ell} e^{-\ell} = \left[E(\ell)\right]_{|r-r'|}^{r+r'}$$

$$U(r) = \int_0^{\infty} dr' \frac{1+f}{2(1+\gamma)} U(r') \left[E(|r-r'|) - E(r+r')\right]$$

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If U (r) is taken to be odd in r this can be represented by

$$U(r) = \int_{-\infty}^{\infty} dr' \; \frac{1+f}{2(1+\gamma)} \; U(r') \, E(|r-r'|)$$

which is formally identical with the plane slab equation. Thus the solution for a sphere is determined by the odd solution in a plane slab of thickness equal to the sphere's diameter.

III. Interior Solution

The character of the solution far from a boundary (where f changes) can be determined by taking the factor $\frac{1+f}{2(1+\gamma)}$ out of the integral and extending the limits to ∞. Then

$$N(x) = \frac{1+f}{2(1+\gamma)} \int_{-\infty}^{\infty} dx' \, N(x') \, E(|x-x'|)$$

for $f > \gamma$ take $N(x) = e^{ikx}$

therefore

$$e^{ikx} = \frac{1+f}{2(1+\gamma)} \int_{-\infty}^{\infty} dx' e^{ikx'} E(|x-x'|) = e^{ikx} \frac{1+f}{2(1+\gamma)} \left[\frac{1}{ik} \ln \frac{1+ik}{1-ik} \right]$$

$$\frac{1+f}{1+\gamma} \; \frac{\tan^{-1} k}{k} = 1$$

for $k \ll 1$, $\quad k \cong \sqrt{\frac{3(f-\gamma)}{1+f}}$

If f and γ are appreciable this result differs considerably from the correct result, more so that the differential diffusion theory result,

$$k_{diff} = \sqrt{3\,(f-\gamma)}$$

K_{diff} differs from K_{int} by a factor which is almost constant for

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constant f. For example, for $f = .5$, we have

$$K_{\varsigma} \cong K_{diff}\,(1.185 + .080\,\gamma)$$

and for $f = .3$

$$K_{\varsigma} \cong K_{diff}\,(1.115 + .083\,\gamma)$$

In general $K_{\varsigma} \doteq k_{diff}\,(1.012 + .343f + .080\gamma)$ to a few thousandths for $.3 \leqslant f \leqslant .7;\ -.2 \leqslant \gamma \leqslant f$.

IV. Boundary Condition.

The diffusion theory boundary condition equates the logarithmic derivatives of the two solutions across a boundary. For a tamped gadget this is a reasonably decent approximation in getting the size of the core although it does not well represent the nature of the solution near the boundary. Untamped, the approximation is quite bad. A much better boundary condition can be obtained by examination of the boundary conditions for those problems which can (so far) be solved exactly.

V. Exact Solution, Untamped Semi-infinite Slab.

The neutron distribution in an untamped or infinitely tamped semi-infinite slab can be obtained by a method patterned after the methods used by Halpern, Luenberg, and Clark and by Uehling in their treatments of the albedo problem.

For untamped semi-infinite slab we have

$$N(x) = \frac{1+f}{2(1+\gamma)} \int_0^{\infty} dx' N(x') E(|x-x'|)$$

write $N(x) = f(x) + g(x)$ where

$$g(x) = 0 \text{ for } x < 0$$

$$f(x) = 0 \text{ for } x > 0$$

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Then

$$g(x) + f(x) = \frac{1+f}{2(1+\gamma)} \int_{-\infty}^{\infty} dx' g(x') E(|x-x'|)$$

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Taking the Laplace transformation of this equation gives, where

$$G(k) = \int_{-\infty}^{\infty} dx\, e^{-kx}\, g(x)$$
$$F(k) = \int_{-\infty}^{\infty} dx\, e^{-kx}\, f(x)$$

$$G(k) + F(k) = \frac{1+f}{2(1+\gamma)} \int_{-\infty}^{\infty} dx\, e^{-kx} \int_{-\infty}^{\infty} dx'\, g(x')\, E(|x-x'|)$$
$$= \frac{1+f}{2(1+\gamma)} \int_{-\infty}^{\infty} dx'\, g(x')\, e^{-kx'} \int_{-\infty}^{\infty} dy\, e^{-ky}\, E(|y|)$$

$y = x - x'$

$$\int_{-\infty}^{\infty} dy\, e^{-ky}\, E(|y|) = \frac{1}{k} \ln \frac{1+k}{1-k}$$

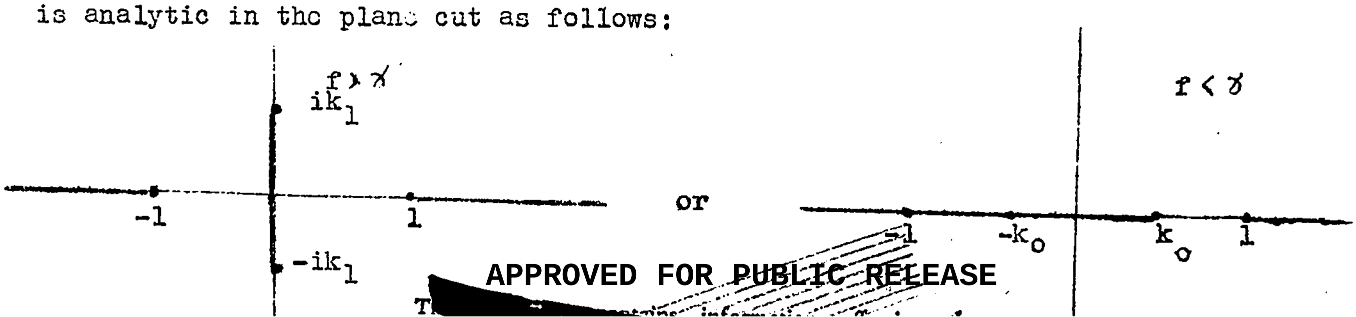
$$G(k) + F(k) = G(k)\, \frac{1+f}{2(1+\gamma)}\, \frac{1}{2k} \ln \frac{1+k}{1-k}$$

$$G(k)\, \rho(k) = G(k) \left[\frac{1+f}{1+\gamma}\, \frac{1}{2k} \ln \frac{1+k}{1-k} - 1 \right] = F(k)$$

$$\ln \rho(k) = \ln F(k) - \ln G(k)$$

F(K) is a linear combination of ascending exponentials hence is analytic in the left half-plane; G(K) is composed of decaying exponentials and is analytic in the right half-plane. Any functions F and G having these analyticity properties such that $\rho G = F$ are solutions of the integral equation of the desired type since the corresponding f(x) and g(x) will vanish right or left respectively.

$\rho(k)$ vanishes at $\pm ik_1$ or $\pm K_o$ according as f is $>$ or $< \gamma$. Then $\ln \rho(k)$ is analytic in the plane cut as follows:



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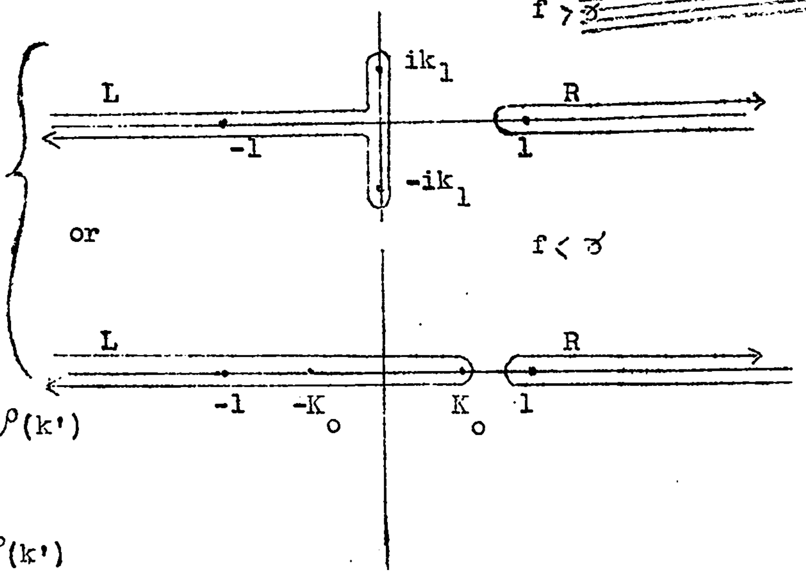
$\ln \rho(k)$ can be represented as



$$\ln \rho(k) = \frac{1}{2\pi i}\int_{C = R+L} \frac{dk'}{k'-k}\ln \rho(k') + \text{const.}$$

$$= \ln \rho_1 + \ln \rho_2 + \text{const.}$$

where

$$\ln \rho_1(k) = \frac{1}{2\pi i}\int_R \frac{dk'}{k'-k}\ln \rho(k')$$

$$\ln \rho_2(k) = \frac{1}{2\pi i}\int_L \frac{dk'}{k'-k}\ln \rho(k')$$

then taking

$$\ln F(k) = \ln \rho_1(k)$$

$$\ln G(k) = -\ln \rho_2(k)$$

satisfies the integral equation and analyticity conditions as $\ln \rho_1$ is analytic to the left, $\ln \rho_2$ to the right. (In the case $f < \gamma$ the solution, $g(x)$, is predominatly the ascending exponential $e^{k_o x}$ so $G(K)$ is, as it should be, analytic only to the right of K_o.

For $f > \gamma$ the contour integral reduces to

$$\ln G(k) = \frac{1}{\pi}\int_0^1 \frac{ds}{s(1-ks)} \tan^{-1}\left[\frac{\pi/2}{\tanh^{-1}S - \frac{1}{S}\,\frac{1+\gamma}{1+f}}\right] - \ln\left[\frac{1+f}{1+\gamma}\,\frac{1}{2R}\ln\frac{1+k}{1-k} - 1\right]$$

The important features of $g(x)$ can be gotten from this expression for $\ln G(K)$ as follows:

$$g(x) = A \sin K_1 (x + x_o) + h(x), \quad h(x) \to 0 \quad \text{as } x \to \infty$$

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then

$$G(k) = \frac{A}{2i}\left[\frac{e^{ik_1x_0}}{k-ik_1} - \frac{e^{-ik_1x_0}}{k+ik_1}\right] + O(1)$$

$$\ell n\, G(\nu + ik_1) = \ell n\, A - \ell n\,(2i) - \ell n\, \nu + ik_1x_0 + \sigma(1)$$

$$\ell n\, G(\nu - ik_1) = \ell n\, A - \ell n\,(-2i) - \ell n\, \nu - ik_1x_0 + \sigma(1)$$

$$2ik_1x_0 = \lim_{\nu \to 0}\left[\ell n G(\nu + ik_1) - \ell n G(\nu - ik_1)\right] + \ell n\,(-1)$$

This limit can be gotten from the above analytic expression for $\ell n\, G(k)$ and gives

$$x_o = \frac{1}{\pi}\int_0^1 \frac{ds}{1 + k_1^2 S^2} \tan^{-1}\left[\frac{\pi/2}{\tanh^{-1} S - \frac{1}{S}\frac{1+\gamma}{1+f}}\right]$$

where as before

$$\frac{\tan^{-1} k_1}{k_1}\;\frac{1+f}{1+\gamma} = 1$$

Here x_o is the distance (as before in units of $\frac{1}{1+\gamma}$ x the mean free path) from the boundary at which the sin function to which the actual distribution is asymptotic vanishes. If the same procedure is followed for the hyperbolic solutions the resulting expression for x_o is of the same form but with $1 + k_1^2 S^2$ replaced by $1 - K_o^2 S^2$ where $\frac{\tanh^{-1} k_o}{k_o}\;\frac{1+f}{1+\gamma} = 1.$

The values of x_o computed from this formula are such as to make the product $x_o \frac{1+f}{1+\gamma}$ very nearly constant, having a minimum value of about .7103 at $\frac{1+f}{1+\gamma} \cong 1.05$ and rising to about .7140 at $\frac{1+f}{1+\gamma} = 1.8$ and to .7152 at $\frac{1+f}{1+\gamma} = .6$. A graph of x_o vs $\frac{1+f}{1+\gamma}$ is given.

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The "offset", x_o, for $f = \gamma = 0$ is .7104 which differs considerably from the Fermi value, $\frac{1}{\sqrt{3}} = .577$.

The value of $g(x)$ at $x = 0$ may be determined by the relation

$$G(k) \cong \frac{g(0)}{k}$$

$$k \gg 1$$

$$\ln g(0) = \lim_{K\to\infty} (\ln G(k) + \ln k)$$

This gives for the linear solution (for $f = \gamma = 0$), $g(x) = .7104 + x$

$g(0) = .5773$ which is, to four significant figures, $\frac{1}{\sqrt{3}}$.

The angular distribution of neutrons emerging from the slab is simply related to $G(k)$.

$$N(\mu) = \mu \int_0^\infty ds\, e^{-s} g(\mu s) = \int_0^\infty dx\, e^{-\frac{x}{\mu}} g(x) = G\left(\frac{1}{\mu}\right)$$

A number of values of $G\left(\frac{1}{\mu}\right)$ have been computed for the linear solution and give a result very closely fitting the Fermi form

$$N(\mu) \propto \mu + \sqrt{3}\,\mu^2$$

VI. Tamped Semi-infinite Slab.

The solution of the tamped slab presents no new difficulties.

The Laplace transformation of the integral equation now takes the form

$$F(k) + G(k) = \left[\frac{1+f}{1+\gamma} G(k) + \frac{1+f_t}{1+\gamma} F(k)\right] \frac{1}{2K} \ln \frac{1+k}{1-k}$$

where f_t is the f for the tamper. It will be zero for a "perfect" tamper, slightly negative for a real tamper (representing absorption.)

Then

$$G(k)\rho(k) = G(k) \frac{1 - \frac{1+f}{1+\gamma} \frac{1}{2k} \ln \frac{1+k}{1-k}}{1 - \frac{1+f_t}{1+\gamma} \frac{1}{2k} \ln \frac{1+k}{1-k}} = -F(k)$$

[redacted]

with regularity conditions as before. $\ln f(k)$ has now branch points [redacted] the six points, ± 1, $\pm ik_1 \left[\frac{\tanh^{-1} k_1}{k_1} \frac{1+f}{1+\gamma} = 1 \right]$ and $\pm k_0 \left[\frac{\tanh^{-1} k_0}{k_0} \frac{1+f_t}{1+\gamma} = 1 \right]$. (Assuming $F > \gamma > f_t$).

The right contour will then enclose 1 and k_0, the left contour -1, $-k_0$, and $\pm ik_1$, as $G(k)$ should be analytic to the right of the imaginary axis, $F(k)$ everywhere left of $+k_0$. The offset of the sinusoidal (core) solution is computed as before, with the result

$$x_0 = \frac{1}{k_1} \tan^{-1} \frac{k_1}{k_0} + \frac{1}{\pi} \int_0^1 \frac{ds}{1+k_1^2 S^2} \left[\tan^{-1} \frac{\pi/2}{\tanh^{-1} S - \frac{1}{S} \frac{1+f}{1+\gamma}} - \tan^{-1} \frac{\pi/2}{\tanh^{-1} S - \frac{1}{S} \frac{1+f_t}{1+\gamma}} \right]$$

where as before

$$\frac{1+f}{1+\gamma} \frac{\tan^{-1} k_1}{k_1} = \frac{1+f_t}{1+\gamma} \frac{\tanh^{-1} k_0}{k_0} = 1$$

The second term is negative and quite small and when divided by $(1+\gamma)$ (in translating the result into mean free paths) is approximately constant. For f in the range .3 to 1.0 and $0 \leq \gamma \leq f$ it has the values .045 $\pm$ .005. The first term alone gives just the diffusion theory boundary condition. Thus a convenient recipe for determining the core radius which is quite accurate over the interesting range is

$$R = .045 + \frac{\pi - \tan^{-1} \frac{k_1}{k_0}}{k_1 (1+\gamma)}$$

[redacted]

where

$$\frac{1+f}{1+\gamma} \frac{\tan^{-1} k_1}{k_1} = \frac{1+f_t}{1+\gamma} \frac{\tanh^{-1} k_0}{k_0} = 1$$

The deviations of the exact solution from its sinusoidal asymptotic form are small and die out very rapidly, about as $e^{-2.5x}$, away from the boundary.* Thus at a distance of one core diameter the discrepancy is negli- [redacted]

*This is true of the straight line solution. For sinusoidal solutions the decay of the deviations is more rapid. This is indicated by the fact that the error in the radius of the untamped sphere determined by this method (checked by a variation method solution) is only 1% of the radius for "zero radius".

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gable and the application of another boundary condition essentially unaffected. Thus the above treatment for the semi-infinite, infinitely tamped slab can be applied without important change to the odd solution in a finite slab, hence to the infinitely tamped sphere.

VII. Anisotrophy of Scattering.

If the scattering is anisotropic then correct results can be obtained from the preceding formulae only if σ_t represents not the total cross section but a judiciously chosen average cross section. For very small k i.e. distributions changing slowly with position, the correct average is the transport average. Since k is not in fact small we must get a better approximation. This will be done by redetermining the relation for k for the infinite sinusoidal solution and using this new k for the scale of length.

If the scattering distribution is

$$\sigma(\mu) = \sigma_0 + \sigma_1 P_1(\mu) + \sigma_2 P_2(\mu) + \ldots..$$

then the neutron distribution,

$$\rho(r,\mu) = e^{ikr} \sum_n \alpha_n P_n(\mu)$$

must satisfy the equation

$$\rho(r,\mu) = \int d\Omega' \int_0^\infty ds\, e^{-(\sigma_0 + \gamma\sigma_0)s} \rho\left(\vec{r} - s\frac{\nu}{|\nu|}, \mu'\right) \left[\frac{\sigma(\mu, \widehat{\mu'}) + (\nu - 1)\sigma_f}{4\pi}\right]$$

(here γ is measured in terms of σ_0)

$$e^{ik.r} \sum_n \alpha_n P_n(\mu) = e^{ik.r} \int d\Omega \int_0^\infty ds\, e^{-(\sigma_0 + \gamma\sigma_0 - ik\mu)s} \times$$

$$\times \sum_n \alpha_n P_n(\mu') \left[\frac{\sum_m \sigma_m P_m(\mu \widehat{,} \mu') + (\nu - 1)\sigma_f}{4\pi}\right]$$

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$$\sum_n \alpha_n P_n(\mu) = \frac{1}{\sigma_0 + \gamma\sigma_0 - ik\mu}\left[\alpha_0\left\{\sigma_0 + (\nu - 1)\sigma_f\right\} P_0 + \sum_{n=1}^{\infty} \frac{\alpha_n P_n(\mu)\,\sigma_n}{2n+1}\right.$$

$$= \frac{1}{\sigma_0 + \gamma\sigma_0 - ik\mu}\sum_{n=0}^{\infty} \frac{\alpha_n \sigma_n P_n(\mu)}{2n+1}\left(1 + f\delta_{no}\right)$$

integrating μ from -1 to 1 with $P_r(\mu)$ gives

$$\frac{2\alpha_n}{2n+1} = \sum_m \frac{\alpha_m \sigma_m}{2m+1}\left(1 + f\delta_{mo}\right)\int_{-1}^{1} d\mu\, \frac{P_n(\mu)\, P_m(\mu)}{\sigma' - ik\mu}$$

where $\sigma' = \sigma_0(1+\gamma)$

writing $\frac{2\alpha_n}{2n+1} = \beta_n$, $\nu = \frac{k}{\sigma'}$, $\tilde{\sigma}_n = \frac{\sigma_n}{\sigma'}$

$$i\nu\beta_n = \sum_{m\leq n}\beta_m\tilde{\sigma}_m\left(1 + f\delta_{mo}\right) P_m\left(\frac{1}{i\nu}\right) Q_n\left(\frac{1}{i\nu}\right)$$

$$+ \sum_{m>n}\beta_m\tilde{\sigma}_m P_n\left(\frac{1}{i\nu}\right) Q_m\left(\frac{1}{i\nu}\right)$$

This can be expressed as a determinantal equation which for assumed values of σ_n, K, and γ gives f. If $\sigma_n = 0$ for n = 1,2,.... the result is just that of the previous treatment. For $\sigma_n = 0$ for n = 2,3,..... the result is

$$\frac{1+\gamma}{1+f} = \frac{\frac{\tan^{-1}\nu}{\nu} - \frac{\tilde{\sigma}_1}{\nu^2}\left(1 - \frac{\tan^{-1}\nu}{\nu}\right)}{1 - \frac{\tilde{\sigma}_1}{\nu^2}\left(1 - \frac{\tan^{-1}\nu}{\nu}\right)}$$

which, in the limit of small ν, gives the diffusion theoretic result involving only the transport cross-section.

With σ_0, σ_1, and σ_2 we have

$$\frac{1+\gamma}{1+f} = \frac{\frac{\tan^{-1}\nu}{\nu} - \frac{\tilde{\sigma}_1}{\nu^2}\left[1 - \frac{\tan^{-1}\nu}{\nu}\right] + \frac{\tilde{\sigma}_1\tilde{\sigma}_2}{4\nu^4}\left[1 - \frac{3}{\tilde{\sigma}}\right]\left[(3+\nu^2)\frac{\tan^{-1}\nu}{\nu} - 3\right]}{1 - \frac{\sigma_1}{\nu^2}\left[1 - \frac{\tan^{-1}\nu}{\nu}\right] + \left[-\frac{\tilde{\sigma}_2}{4\nu^4}(3+\nu^2) + \frac{\tilde{\sigma}_1\tilde{\sigma}_2}{4\nu^4}\right]\left[(3+\nu^2)\cdots\right.}$$

[redacted]

A three term expansion fitting reasonably well present knowledge of the cross sections gives a critical radius two per cent larger than that derived from the isotropic scattering formula using the transport average for the scattering cross section. This discrepancy increases to three per cent at about two critical radii.

Combining this result with the integral correction formula gives,

$$R = \left[.045 + \frac{\pi - \tan^{-1} \frac{k_1}{k_o}}{k_1} \right] \lambda_{trans}$$

$$k_1 = \sqrt{3(f - \gamma)}\ (.99 + .34f + .05\gamma)$$

$$k_o = \sqrt{3\ (\gamma - f_t)}\ (.99 + .34\ f_t + .05\gamma)$$

where $f = \dfrac{(\nu - 1)\sigma_f}{\sigma_{trans}}$, γ defined by $e^{\gamma t \nu N \sigma_{trans}}$ time dependence.

The appended curves give the untamped extrapolated end point, x_o, as a function of $\frac{1 + f}{1 + \gamma}$ and the approximate shape of N(x) for $\frac{1 + f}{1 + \gamma} = 1$ for the untamped half-infinite slab.

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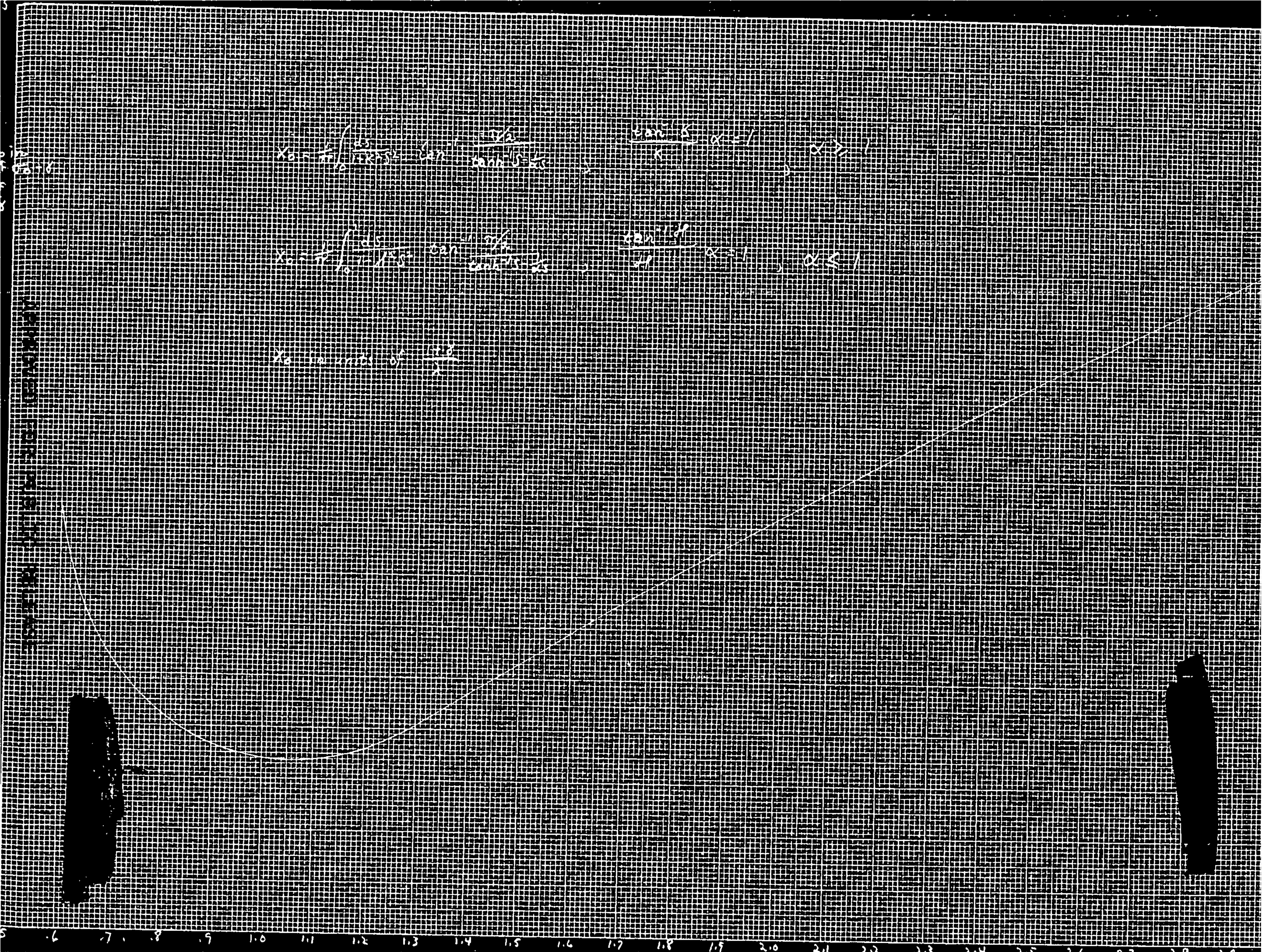

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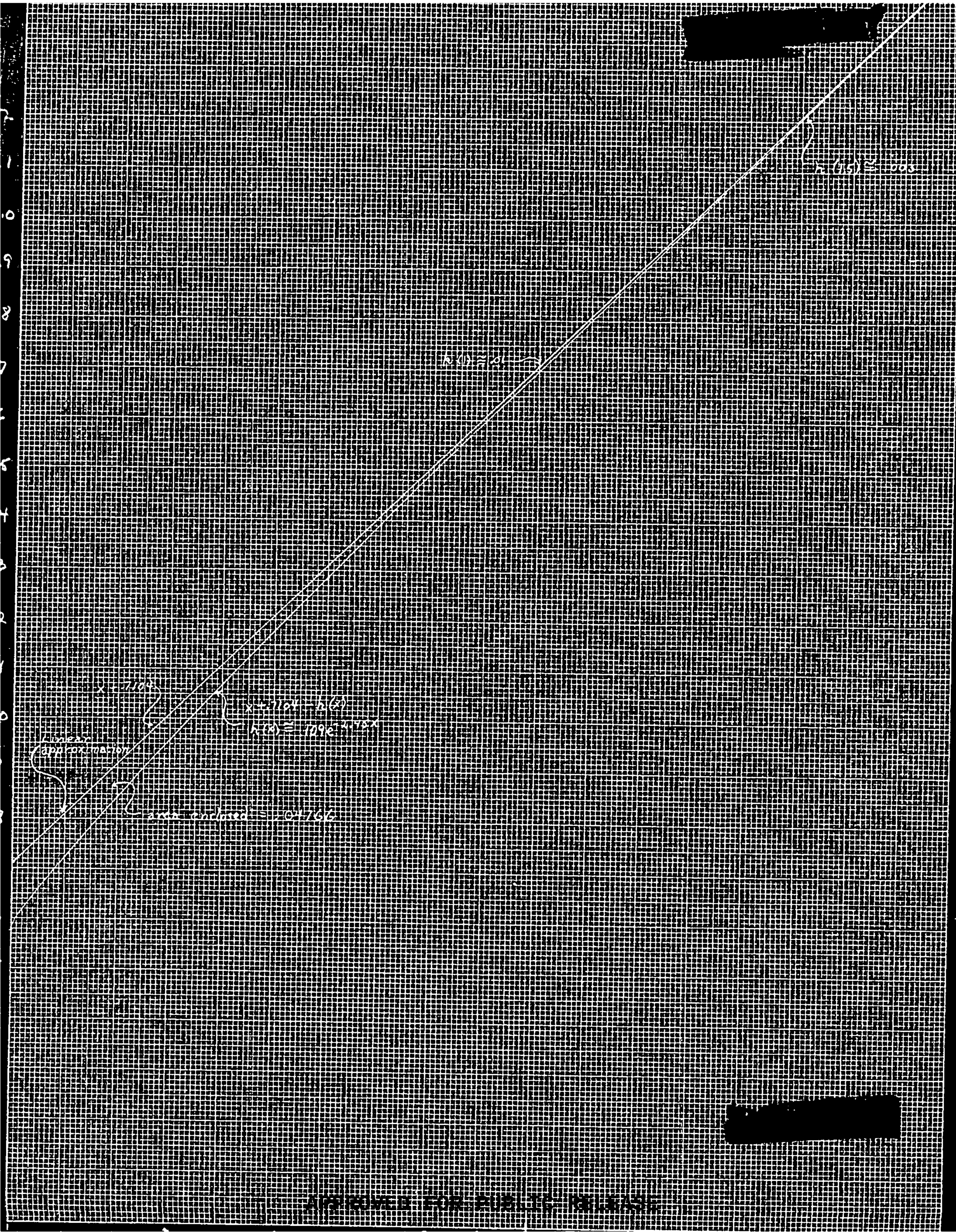

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